Relating the Trace and Frobenius Matrix Norms

Jason D. M. Rennie
jrennie@gmail.com

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Among other things, Srebro [1] discusses two matrix norms: the trace norm and the Frobenius norm. The trace norm of a matrix X is defined as the L_1 norm of the singular values of X. Let $X = U\Lambda V^T$ be the singular value decomposition of X, and let $\vec{\lambda} = \mathrm{diag}(\Lambda)$ be the vector of singular values (the diagonal entries of Λ). Then,

$$\|X\|_{\mathrm{tr}} = \|\vec{\lambda}\|_1 = \sum_i |\lambda_i|. \tag{1}$$

The Frobenius norm of a matrix X is the L_2 norm of the vector of singular values,

$$\|X\|_{\mathrm{Fro}} = \|\vec{\lambda}\|_2 = \sqrt{\sum_i \lambda_i^2}. \tag{2}$$

Srebro states the following Lemma,

Lemma 1 *For any matrix X, $\|X\|_{Fro} \le \|X\|_{tr} \le \sqrt{rankX}\|X\|_{Fro}$, where rank$(X)$ is the number of non-zero singular values of X.*

A brief proof is given. We find the proof satisfactory for establishing the left inequality, but feel that additional explanation is helpful for establishing the right inequality.

Consider a matrix X with $\mathrm{rank}X = k > 0$ and $\|X\|_{\mathrm{tr}} = t$. Now consider finding the length k vector, $\vec{x}$, with $\sum_i x_i = t$ such that the L_2 norm is minimized. We show that the minimum such vector, $\vec{y}$, has $y_i = t/k\ \forall i$. Consider any vector $\vec{x}$ that satisfies the constraints. Then $\vec{y}$ is a convex combination of permutations of $\vec{x}$. As $f(x) = x^2$ is a concave function, $\|\vec{x}\|_2 \ge \|\vec{y}\|_2$. Note that $\|\vec{y}\|_2 = t/\sqrt{k}$. Hence, $\|\vec{\lambda}\|_1 \le \|\vec{\lambda}\|_2\sqrt{\mathrm{rank}X}$.

References

[1] N. Srebro. *Learning with Matrix Factorizations*. PhD thesis, Massachusetts Institute of Technology, 2004.