The Relation Between the Spectral and Trace Norms

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According to Mathworld [2], the spectral norm of a matrix is the natural norm induced by the L2-norm,

$$\|A\| = \max_{\|\vec{x}\|_2=1} \|A\vec{x}\|_2, \tag{1}$$

where $\|\cdot\|_2$ denotes the L2-norm. The trace norm[1] of a matrix X, denoted $\|X\|_\Sigma$, is the sum of singular values of a matrix. Fazel et al. [1] note the the spectral and trace norms are dual to each other. The trace norm can be defined as the maximum over matrices of spectral norm 1 or less of the trace of a matrix product,

$$\|X\|_\Sigma = \max_{Y:\|Y\|\leq 1} \text{tr}(Y^T X), \tag{2}$$

where $\text{tr}(\cdot)$ denotes trace of the argument. The duality allows us to define the spectral norm similarly,

$$\|X\| = \max_{Y:\|Y\|_\Sigma\leq 1} \text{tr}(Y^T X). \tag{3}$$

References

[1] M. Fazel, H. Hindi, and S. P. Boyd. A rank minimization heuristic with application to minimum order system approximation. In *Proceedings of the American Control Conference*, volume 6, pages 4734–4739, 2001.

[2] E. W. Weisstein. Spectral norm. http://mathworld.wolfram.com/SpectralNorm.html. From MathWorld–A Wolfram Web Resource.

[1] The trace norm is also known as the nuclear norm and the Ky-Fan n-norm.