Whence the Determinant?

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Abstract

We provide some background on determinants. This discussion is not particularly formal. See Halmos [1] for a more careful discussion.

According to MathWorld, a vector space is "a set that is closed under finite addition and scalar multiplication." [4] A bilinear form is a function from two vector spaces to the reals, $f : V \times V \to \mathbb{R}$, which is linear in each of its arguments [2],

$$f(\alpha\vec{v}, \vec{w}) = f(\vec{v}, \alpha\vec{w}) = \alpha f(\vec{v}, \vec{w}) \tag{1}$$
$$f(\vec{v}_1 + \vec{v}_2, \vec{w}) = f(\vec{v}_1, \vec{w}) = f(\vec{v}_2, \vec{w}) \tag{2}$$
$$f(\vec{v}, \vec{w}_1 + \vec{w}_2) = f(\vec{v}, \vec{w}_1) = f(\vec{v}, \vec{w}_2) \tag{3}$$

A multilinear form (also called an n-linear form) generalizes the bilinear form to more than two arguments, $g : V \times \cdots \times V \to \mathbb{R}$; a multilinear form is linear in each of its arguments,

$$g(\vec{v}_1, \dots, \alpha\vec{v}_k, \dots, \vec{v}_n) = \alpha g(\vec{v}_1, \dots, \vec{v}_k, \dots, \vec{v}_n), \forall k, \tag{4}$$
$$g(\vec{v}_1, \dots, \vec{v}_k + \vec{v}'_k, \dots, \vec{v}_n) = g(\vec{v}_1, \dots, \vec{v}_k, \dots, \vec{v}_n) + g(\vec{v}_1, \dots, \vec{v}'_k, \dots, \vec{v}_n), \tag{5}$$

$\forall k$ [3]. A skew-symmetric multilinear form is a multilinear form such that the sign changes when two adjacent arguments are swapped (§30 of [1]). Let h be a multilinear form such that

$$h(\dots, \vec{v}_i, \vec{v}_{i+1}, \dots) = -h(\dots, \vec{v}_{i+1}, \vec{v}_i, \dots) \quad \forall i. \tag{6}$$

Then, h is a skew-symmetric multilinear form. An alternating multilinear form is a multilinear form that returns zero if two arguments are identical (§30 of [1]). Let l be a multilinear form such that for any $\vec{v}$,

$$l(\dots, \vec{v}, \dots, \vec{v}, \dots) = 0. \tag{7}$$

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Note that since l is a multilinear form, l must be zero whenever one argument is a linear function of the other arguments,

$$l(\vec{v}_1, \ldots, \vec{v}_{n-1}, \sum_{i=1}^{n-1} \alpha_i \vec{v}_i) = \sum_{i=1}^{n-1} \alpha_i l(\vec{v}_1, \ldots, \vec{v}_{n-1}, \vec{v}_i) = 0. \quad (8)$$

It can be generally shown that every alternating mulilinear form is skew-symmetric (§30 of [1]); the converse (that every skew-symmetric multilinear form is alternating) is not generally true, but it is true when $V \equiv \mathbb{R}^n$.

Halmos (§31 of [1]) shows that the vector space of alternating n-linear forms on an n-dimensional vector space is one-dimensional. As a result, if we define an alternating n-linear form as

$$f(\vec{v}_1, \ldots, \vec{v}_n) = g(A\vec{v}_1, \ldots, A\vec{v}_n), \quad (9)$$

where A is a linear transform on an n-dimensional vector space, then

$$f(\vec{v}_1, \ldots, \vec{v}_n) = \delta_A g(\vec{v}_1, \ldots, \vec{v}_n), \quad (10)$$

where δ_A is a scalar (§53 of [1]). δ_A is called the determinant of A. A bit more work reveals that the determinant is the unique (up to a scalar multiple) alternating multilinear form.

Many thanks to John Barnett for explaining to me these concepts.

References

[1] P. R. Halmos. *Finite-Dimensional Vector Spaces.* Litton Educational Publishing, 1958.

[2] E. W. Weisstein. Bilinear form. http://mathworld.wolfram.com/BilinearForm.html. From MathWorld–A Wolfram Web Resource.

[3] E. W. Weisstein. Multilinear form. http://mathworld.wolfram.com/MultilinearForm.html. From MathWorld–A Wolfram Web Resource.

[4] E. W. Weisstein. Vector space. http://mathworld.wolfram.com/VectorSpace.html. From MathWorld–A Wolfram Web Resource.